JOURNAL OF APPROXIMATION THEORY 39, 202–210 (1983)

A Geometric Characterization of H-Sets

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Communicated by Oved Shisha

Received October 22, 1979; revised August 2, 1982

The concept of H-sets with respect to a finite-dimensional linear space of approximation is extremely important, as these sets unify the theory of Chebyshev approximation by yielding a characterization of best approximants and conditions for uniqueness, and are also aids in the construction of algorithms for numerical computation. We study here the geometric characterization of H-sets in terms of convex polyhedral cones or, in special cases, simplices. In particular we consider the characterization of minimal H-sets and use this to find an upper bound for the number of possible minimal H-sets with respect to a finite-dimensional space.

INTRODUCTION

The importance of H-sets in the study of best Chebyshev approximation has been highlighted in [1–5, 7–9, 12]. The characterization of best approximants was discussed in [1, 3–5, 8] and a study was made in [2, 9] of the problem of finding the set of best approximants in the non-uniqueness case. Using the characterization, an algorithm was suggested in [2] for computing a best approximation to a continuous function by a space of functions not satisfying the Haar condition.

The construction of H-sets for particular linear spaces has been studied in [5–7], where attention has been focused on the multivariate problem. In [6, 7] some topological properties are given of H-sets with respect to the space of polynomials of degree at least m in 2 variables.

Here we consider the problem of characterizing H-sets geometrically. Although such a characterization for a particular type of space of real-valued functions was made in [11], here we extend this concept to general H-sets with respect to a linear subspace of functions, devoting special attention to minimal H-sets. We characterize minimal H-sets in terms of convex polyhedral cone intersections and arrive at an upper bound for the number of possible minimal H-sets with respect to an n-dimensional subspace.

Although most problems in approximation theory concern continuous

0021-9045/83 $3.00

real- or complex-valued functions, we consider the more general setting of functions with values in some normed linear space. In this general setting the same characterization of best approximation exists and also similar theorems concerning uniqueness, strong unicity, and maximal linear functionals (see [3]). For general definitions in topological vector spaces we follow [10].

DEFINITIONS

Let X be a compact topological space and Y a Banach space over the scalar field $K = \mathbb{R}$ or $\mathbb{C}$. We denote by $C_K(X, Y)$ the space of continuous functions defined on X with values in Y over the same scalar field K. We denote by Y^* the dual space of Y consisting of all continuous linear functionals defined on Y. We define by $\mathscr{S}$ the unit sphere in Y^*, i.e., $\mathscr{S} = \{l \in Y^*: \|l\| = 1\}$, where the norm is the usual operator norm in Y^*.

We consider H-sets with respect to a finite-dimensional linear subspace V of $C_K(X, Y)$ which has $g_1, \ldots, g_n$ as a basis. If $K = \mathbb{C}$, we extend this basis to one of dimension $2n$ over $\mathbb{R}$ in the usual fashion. We denote by $\mathbb{R}^k_+$ the positive orthant of $\mathbb{R}^k$ and denote by θ_n the zero vector in $\mathbb{R}^n$.

DEFINITION 1. The set of k points $x_1, \ldots, x_k$ in X together with k elements $l_1, \ldots, l_k$ of $\mathscr{S}$ form an *H-set with respect to V*, denoted by $[\{x_i\}, \{l_i\}, k]$, if and only if there exists $\lambda \in \mathbb{R}^k_+$ such that

$$\mathscr{M}\lambda = \theta_n,$$

where $\mathscr{M}_{ij} = l_j[g_i(x_j)]$, $j = 1, \ldots, k$, $i = 1, \ldots, n$.

DEFINITION 2. The H-set $[\{x_i\}, \{l_i\}, k]$ is a *minimal H-set with respect to* $[V, \{l_i\}]$ if no proper subset of $[\{x_i\}, \{l_i\}, k]$ can form an H-set with respect to V.

This definition of minimal H-set is dependent on the set $\{l_i\}$ and it is certainly feasible that there exists a set $\{l_i'\}$ in $\mathscr{S}$ such that $[\{x_i\}, \{l_i'\}, k]$ is an H-set with respect to V and not minimal with respect to $[V, \{l_i'\}]$. However, we can deduce the following:

LEMMA 1. *$[\{x_i\}, \{l_i\}, k]$ is a minimal H-set with respect to $[V, \{l_i\}]$ if and only if there exists only one solution, up to a scalar multiple, to $\mathscr{M}\lambda = \theta_n$, with $\lambda \in \mathbb{R}^k_+$.*

Proof. Suppose $[\{x_i\}, \{l_i\}, k]$ is a minimal H-set with respect to $[V, \{l_i\}]$. From the definition of an H-set there exists a $\lambda \in \mathbb{R}^k_+$ such that $\mathscr{M}\lambda = \theta_n$.

Let $\mu \in \mathbb{R}^k$ be such that $\mathcal{M}\mu = \theta_n$ and choose $p = \min_i\{-\lambda_i/\mu_i : \mu_i \neq 0\}$ then $\lambda + p\mu \in \mathbb{R}^k_+$, $\mathcal{M}(\lambda + p\mu) = \theta_n$ and at least one component of $\lambda + p\mu$ is zero. Hence if $-\lambda \neq p\mu$, then $|\{x_i\}, \{l_i\}, k|$ is not minimal with respect to $|V, \{l_i\}|$.

Conversely, if there exists a $\lambda \in \mathbb{R}^k_+$ such that $\mathcal{M}\lambda = \theta_n$ then $|\{x_i\}, \{l_i\}, k|$ is an H-set with respect to V. If there is only one such solution then no component of λ can be made zero. Hence $|\{x_i\}, \{l_i\}, k|$ is minimal with respect to $|V, \{l_i\}|$.

An important feature of this lemma is

Corollary 1. *If for* $|\{x_i\}, \{l_i\}, k|$ *there is a* $\lambda \in \mathbb{R}^k_+$ *and* $\mu \in \mathbb{R}^k$ *such that* $\mathcal{M}\lambda = \mathcal{M}\mu = \theta_n$ *and* μ *is not a simple multiple of* λ *then* $|\{x_i\}, \{l_i\}, k|$ *is not a minimal H-set with respect to* $|V, \{l_i\}|$.

It is also immediate from the lemma that

Corollary 2. *If* $Y = K$ *and* $|\{x_i\}, \{l_i\}, k|$ *is a minimal H-set with respect to* $|V, \{l_i\}|$ *then every choice of* $\{l'_i\}$, *such that* $|\{x_i\}, \{l'_i\}, k|$ *is an H-set with respect to V, makes this H-set minimal.*

Definition 3. Define the operator $\mathcal{F}$, mapping elements $x_i \in X$ of the H-set $M = |\{x_i\}, \{l_i\}, k|$ with respect to V into K^n, such that

$$\mathcal{F}x_i = (l_i|g_1(x_i)|, \ldots, l_i|g_n(x_i)|).$$

For brevity we write $\mathcal{F}(M)$ for the range of $\mathcal{F}$ with domain $\{x_i\}$ of M.

Definition 4. The subsets $M_1 = |\{x_i^{(1)}\}, \{l_i^{(1)}\}, k_1|$ and $M_2 = |\{x_i^{(2)}\}, \{l_i^{(2)}\}, k_2|$ form an H-partition $|M_1, M_2|$ of the H-set $M = M_1 \cup M_2$ with respect to V if and only if

(a) neither M_1 nor M_2 are void;

(b) $M_1 \cap M_2 = \varnothing$ (empty set);

(c) neither M_1 nor M_2 form an H-set with respect to V.

Definition 5. An H-partition $|M_1, M_2|$ is canonical if and only if there exists a $\lambda \in \mathbb{R}^k_+$, a solution to $\mathcal{M}\lambda = \theta_n$ for the H-set $M = M_1 \cup M_2$ with respect to V, such that

$$\sum_{i \in I_1} \lambda_i = \sum_{i \in I_2} \lambda_i,$$

where $I_p = \{i : x_i \in M_p\}$, $p = 1, 2$.

The convex hull of a set A we shall denote by $\mathscr{H}(A)$ and the convex polyhedral cone of a finite set A by $\mathscr{C}(A)$; thus

$$\mathscr{H}(A) = \left\{ \sum \lambda_i a_i : a_i \in A, \lambda_i > 0 \text{ and } \sum \lambda_i = 1 \right\},$$

$$\mathscr{C}(A) = \left\{ \sum \lambda_i a_i : a_i \in A, \lambda_i > 0 \right\}.$$

The points of a finite set $A = \{a_1, ..., a_r\}$ are pointwise independent if the $r-1$ elements $\{a_2 - a_1, ..., a_r - a_1\}$ are linearly independent. For such a set A, $\mathscr{H}(A)$ is a simplex of order $r-1$.

Characterization

The characterization of H-sets in the framework of these concepts follows directly.

THEOREM 1. *If* $[M_1, M_2]$ *is an* H*-partition of the* H*-set* $M = [\{x_i\}, \{l_i\}, k]$ *with respect to* V, *then* $\mathscr{C}(\mathscr{F}(M_1)) \cap \mathscr{C}(-\mathscr{F}(M_2))$ *is not empty.*

Conversely, given $M_1 = [\{x_i^{(1)}\}, \{l_i^{(1)}\}, k_1]$ *and* $M_2 = [\{x_i^{(2)}\}, \{l_i^{(2)}\}, k_2]$ *with* $M_1 \cap M_2 = \varnothing$, *if* $\mathscr{C}(\mathscr{F}(M_1)) \cap \mathscr{C}(-\mathscr{F}(M_2))$ *is not empty, then* $M = M_1 \cup M_2$ *is an* H*-set with respect to* V.

Proof. From Definition 1 there are positive real values $\lambda_1, ..., \lambda_k$ such that

$$\sum_{I_1} \lambda_i \mathscr{F}(x_i) = \sum_{I_2} \lambda_i(-\mathscr{F}(x_i)),$$

where $I_p = \{i : x_i \in M_p\}$, $p = 1, 2$; this defines a ray in $\mathscr{C}(\mathscr{F}(M_1)) \cap \mathscr{C}(-\mathscr{F}(M_2))$. This argument is reversed to prove the converse.

THEOREM 2. *Given the conditions as in Theorem* 1, $[M_1, M_2]$ *forms a canonical* H*-partition of* $M = M_1 \cup M_2$ *if and only if* $\mathscr{H}(\mathscr{F}(M_1)) \cap \mathscr{H}(-\mathscr{F}(M_2))$ *is not empty.*

Proof. The proof is the same as for Theorem 1 except that in this case we can choose the λ_i, $i = 1, ..., k$, such that

$$\sum_{I_1} \lambda_i = \sum_{I_2} \lambda_i = 1.$$

We note that not all H-sets have a canonical H-partition, as the following counterexample using real-valued functions shows.

EXAMPLE 1. Let $V = \operatorname{span}\{x, y, z\}$; we have an H-set $[\{p_i\}, \{l_i\}, 3]$ with respect to V where

$$p_1 = (1, 0, -1); \qquad l_1 = -1.$$

$$p_2 = (0, 1, -1); \qquad l_2 = -1.$$

$$p_3 = (1, 1, -2); \qquad l_3 = +1.$$

The solution to $\mathscr{M}\lambda = \theta_3$ is $(1, 1, 1)$ which does not have a canonical decomposition.

Theorem 2 is an extension of the characterization given in [11], where it was assumed that the functions in V were real-valued and the constant function was an element of V. In such a case a canonical H-partition always exists. We note that our proof is simpler than that given in [11] for this case.

MINIMAL H-SETS

The important applications of Theorems 1 and 2 arise when minimal H-sets are considered. The characterization of minimal H-sets allows us to find how many possible such sets can be constructed, thus answering the question posed in [11]. A simple observation concerning minimal H-sets is the following.

THEOREM 3. *$M = [\{x_i\}, \{l_i\}, k]$ forms a minimal H-set with respect to $[V, \{l_i\}]$ if and only if no $x_p \in \{x_i\}$ exists such that $\mathscr{F}(x_p)$ is a linear combination of less than $k-1$ vectors $\mathscr{F}(x_i)$, $i \neq p$.*

Proof. Without loss of generality suppose

$$\mathscr{F}(x_k) = \sum_1^r \alpha_i \mathscr{F}(x_i), \qquad r < k-1.$$

From Definition 1 we have that there exist positive λ_i, $i = 1, \ldots, k$, such that

$$\sum_1^k \lambda_i \mathscr{F}(x_i) = \theta_n.$$

From our supposition and this equation we have

$$\lambda_k \sum_1^r \alpha_i \mathscr{F}(x_i) + \sum_1^{k-1} \lambda_i \mathscr{F}(x_i) = \theta_n$$

and as $r < k-1$ the coefficients of $\mathscr{F}(x_i)$, $i = 1,\dots,k-1$, cannot all be zero. Hence from the Corollary 1 of Lemma 1, $[\{x_i\}, \{l_i\}, k]$ is not minimal with respect to $[V, \{l_i\}]$ and the result follows.

Conversely, suppose M is not minimal with respect to $[V, \{l_i\}]$, then we have two solutions $\lambda_1,\dots,\lambda_k$ and $\mu_1,\dots,\mu_k$ such that $\mathscr{M}\lambda = \mathscr{M}\mu = \theta_n$, λ, $\mu \in \mathbb{R}^k_+$ and $\lambda \neq p\mu$ for any scalar p, hence the ratio of least two components of λ and μ are different, thus we assume $\lambda_k/\mu_k \neq \lambda_{k-1}/\mu_{k-1}$. Hence

$$-\lambda_k \mathscr{F}(x_k) = \sum_1^{k-1} \lambda_i \mathscr{F}(x_i),$$

$$-\mu_k \mathscr{F}(x_k) = \sum_1^{k-1} \mu_i \mathscr{F}(x_i);$$

therefore

$$\mathscr{F}(x_k) = \sum_1^{k-2} (\mu_{k-1}\lambda_i - \lambda_{k-1}\mu_i)\, \mathscr{F}(x_i)/(\mu_k\lambda_{k-1} - \lambda_k\mu_{k-1})$$

and the result follows.

In terms of our Characterization Theorem 1 we have the following.

THEOREM 4. *Let* $M_1 = [\{x_i\}, \{l_i\}, i = 1,\dots, r]$ *and* $M_2 = [\{x_i\}, \{l_i\}, i = r+1,\dots, k]$ *such that* $M_1 \cap M_2 = \varnothing$*: then* $M = M_1 \cup M_2$ *is a minimal* H*-set with respect to* $[V, \{l_i\}]$ *if and only if* $U = \mathscr{C}(\mathscr{F}(M_1)) \cap \mathscr{C}(-\mathscr{F}(M_2))$ *consists of a ray which does not lie on a face of either cone when* $1 < r < k-1$.

Proof. We first prove that U consists of a single ray. Suppose in U we have two distinct rays defined by

$$\sum_1^r \lambda_i \mathscr{F}(x_i) \quad \text{and} \quad \sum_1^r \mu_i \mathscr{F}(x_i) \quad \text{in } \mathscr{C}(\mathscr{F}(M_1)),$$

$$\sum_{r+1}^k \lambda_i(-\mathscr{F}(x_i)) \quad \text{and} \quad \sum_{r+1}^k \mu_i(-\mathscr{F}(x_i)) \quad \text{in } \mathscr{C}(-\mathscr{F}(M_2)).$$

As these rays are distinct, the vectors $\lambda = (\lambda_1,\dots,\lambda_k)$ and $\mu = (\mu_1,\dots,\mu_k) \in \mathbb{R}^k_+$ and are solutions to $\mathscr{M}\lambda = \mathscr{M}\mu = \theta_n$ such that $\lambda \neq \alpha\mu$ for any α, contradicting the minimality of M.

Conversely, if such λ and μ exist we can define two rays.

To show that this single ray is interior to each cone we use Theorem 3. If the ray lay on the face of $\mathscr{C}(\mathscr{F}(M_1))$, which without loss of generality we assume to be generated by $A = \{\mathscr{F}(x_i): i = 2,\dots, r\}$, then $\mathscr{F}(x_1)$ would be a linear combination of A, contradicting Theorem 3. Similarly for $\mathscr{C}(-\mathscr{F}(M_2))$.

For a consideration of H-sets where a canonical H-partition exists, we need the following.

THEOREM 5. *Let* $M = [\{x_i\}, \{l_i\}, k]$ *be a minimal H-set with respect to* $[V, \{l_i\}]$ *and* $N = [\{x_i\}, \{l_i\}, i = 1,..., r]$, $r < k$; *then* $\mathscr{E}(N)$ *forms a pointwise independent set.*

Proof. First consider the case when $r < k$.

Let $p_i = \mathscr{E}(x_i) - \mathscr{E}(x_1)$, $i = 2,..., r$, and suppose these p_i are linearly dependent. We then have α_i, $i = 2,..., r$, not all zero such that

$$\sum_2^r \alpha_i p_i = \theta_n.$$

If we define

$$\alpha_1 = -\sum_2^r \alpha_i,$$

then

$$\sum_2^r \alpha_i p_i = \sum_1^r \alpha_i \mathscr{E}(x_i) = \theta_n$$

and hence the vector $(\alpha_1,..., \alpha_r, 0,...) \in \mathbb{R}^k$ is a solution to $\mathscr{M}\alpha = \theta_n$, which contradicts the minimality of M from Corollary 1 of Lemma 1.

For the case when $r = k$ there does exist a solution to

$$\sum_1^k \alpha_i \mathscr{E}(x_i) = \theta_n,$$

where $\alpha_i > 0$. For p_i, $i = 2,..., r$, as above to be linearly independent, α_1 must be chosen similarly to α_1 above. This makes $\alpha_1 < 0$, a contradiction.

We can now consider the case of H-sets with a canonical H-partition.

THEOREM 6. *Let* $M_1 = [\{x_i^{(1)}\}, \{l_i^{(1)}\}, k_1]$ *and* $M_2 = [\{x_i^{(2)}\}, \{l_i^{(2)}\}, k_2]$, $M_1 \cap M_2 = \phi$. *Then* $M = M_1 \cup M_2$ *is a minimal H-set with respect to* $[V, \{l_i\}]$ *with canonical H-partition* $[M_1, M_2]$ *if and only if* $\mathscr{H}(\mathscr{E}(M_1))$ *and* $\mathscr{H}(-\mathscr{E}(M_2))$ *are simplices, whose intrsection consists of a single vector interior to both simplices.*

Proof. The fact that they are simplices follows from Theorem 5. The single vector interior to both simplices is that vector where the single ray $\mathscr{C}(\mathscr{E}(M_1)) \cap \mathscr{C}(-\mathscr{E}(M_2))$ of Theorem 4 passes through the simplices.

We can now consider the question, originally posed in [11], of how many possible minimal H-sets with respect to V can there be. A study was made in [12] for the particular case of the $(n+1)$-dimensional space $V = \text{span}\{1, x_1,..., x_n\}$ with $x_i \in \mathbb{R}$. There it was found that the minimal H-sets with respect to V could be enumerated, and the precise number was given. We note here that neither the degenerate case with $x_i = 0$ for all i is possible due to the constant function being an element of V, nor can a minimal H-set with two points be formed for the same reason. Both of these cases are possible if the constant function is not an element of V, as is the case, for example, for $V = \text{span}\{x, y\}$. We are, therefore, able to obtain the following value for the maximum number of possible minimal H-sets. We note that this value is attained for the case of $V = \text{span}\{x_1, x_2, \dots, x_n\}$ with $x_i \in \mathbb{R}$. On the other hand, for approximating sets satisfying the Haar condition there can only be one minimal H-set.

THEOREM 7. *Let V be a space of dimension n; then at most $h(n)$ minimal H-sets with respect to V are possible where*

$$\begin{aligned} h(n) &= k^2 + k + 1 \qquad \textit{if} \quad n = 2k, \\ &= k^2 + 2k + 2 \qquad \textit{if} \quad n = 2k + 1. \end{aligned}$$

Proof. We first note that from Definition 1 the maximum number of points j in a minimal H-set $[\{x_i\}, \{l_i\}, k]$ with respect to $[V, \{l_i\}]$ is $n + 1$.

Suppose a minimal H-set $[\{x_i\}, \{l_i\}, k]$ with respect to $[V, \{l_i\}]$ does not have a canonical H-partition. We can for this case construct a diagonal matrix D, with diagonal $(1, \dots, 1, \alpha, 1, \dots, 1)$, such that

$$\mathscr{A} D^{-1} D \lambda = \theta_n,$$

giving a minimal H-set with respect to $[V, \{l_i\}]$ and having a canonical H-partition.

We say that all minimal H-sets with respect to $[V, \{l_i\}]$ are basically the same, and, therefore, only count them once, if they have the same p, q where $[M_p, M_q]$ is a canonical H-partition of the H-set or projected canonical H-partition as above for the non-canonical case. For reasons of symmetry we presume $p \geqslant q$.

We are thus led to the conclusion that the number of basically different minimal H-sets of size j is given by the number of pairs p, q such that $p \geqslant q \geqslant 1$ and $p + q = j$. This value is $[(j-2)/2] + 1$, where $[\ ]$ denotes the greatest integer. If we add on the possibility of the degenerate case mentioned above then $h(n)$ must be

$$1 + \sum_{j=2}^{n+1} \left\{ \left[\frac{j-2}{2} \right] + 1 \right\},$$

which gives the result as stated in the theorem.

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Printed in Belgium